Calculating Concentration-Sensitive Capital Charges with Conditional Value-at-Risk

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Abstract. By the end of 2003, the Basel Committee on Banking Supervision
(BCBS) will launch its final recommendations on minimum capital requirements
in the banking industry. Although there is the intention to arrive at capital charges
which concur with economic intuition, the risk weight formulas proposed by the
committee will lack an adequate treatment of concentration risks in credit portfo-
lios. The question arises whether this problem can be solved without recourse to
fully-fledged portfolio models. Since recent practical experience shows that the risk
measure Conditional Value-at-Risk (CVaR) is particularly well suited for detecting
concentrations, we develop the semi-asymptotic approach by Emmer and Tasche in
the CVaR context and compare it with the capital charges recently suggested by
the Basel Committee. Both approaches are based on the same Vasicek one-factor
model.

1 Introduction

From an economic point of view, the risks that arise in a portfolio need to be
covered by a corresponding amount of capital that is applied as a cushion to
absorb potential losses. The question of calculating the risk contributions of
single assets in a portfolio corresponds to the problem of calculating capital
charges to cover occurring loss risks. This allocation issue can be considered
as well from a regulatory as well as from an internal perspective, leading to
capital charges per asset by the regulatory and the economic capital, respec-
tively.

Considering the different regulations for credit risk, we observe that in the
current regulatory regime (Basel I Accord) almost no risk adjustment can be
identified. By the end of 2003, the BCBS will launch its final recommenda-
tions on new minimum capital requirements in the banking industry (Basel II
Accord). Although there is the intention to arrive at improved more risk sen-
sitive capital charges of the credit risk bearing assets, the risk weight formulas
proposed by the committee will lack an adequate treatment of concentration
risks in credit portfolios as we will show below. The main problem is that
the Basle II model assumes that all credits are of equal, infinitesimal small

exposure, i.e. that the credit portfolio is infinitely fine grained. However, in real world portfolios, this basic assumption does not hold.

From an internal perspective, banks are putting high efforts into the development of internal credit risk models that allow the risk measurement of the portfolio credit risk. Comprehensive research work has been done to develop methods of how to calculate risk contributions in an appropriate way (see e.g. [4], [7], [9], [11], [12]). However, from a practical point of view, the allocation problem, i.e. the question of how the single assets contribute to the overall portfolio risk, cannot yet be considered as solved. Risks are broken down in different ways, taking into account correlation effects and concentration risk to different extents (see e.g. [3]). Banks that are using internal credit risk models in most cases need enormous calculation efforts to estimate the overall portfolio risk and the risk contributions of single assets and sub-portfolios.

In this paper, we give a survey on an approach to calculate risk contributions of single assets in an analytical way that avoids calculation intensive simulation efforts. This *semi-asymptotic* approach slightly extends the Basel II approach and takes into account concentration effects. Thus, it can be viewed as a bridging to relate regulatory and internal risk measurement. Additionally, it is based on the new risk measure of Conditional Value at Risk that has been proven to be appropriate for bank wide loss risk measurement (see for instance [1], [10], [12]).

The paper is organized as follows. In Section 2 we briefly introduce the capital charges as they are suggested by the Basel Committee. Section 3 presents the semi-asymptotic approach to capital charges in case of CVaR as risk measure. In Section 4 we illustrate the both approaches with a numerical example.

2   The Basel II Model

We give a short presentation of the reasoning that lead to the current suggestions by the Basel Committee. In particular, we introduce the so-called Vasicek one-factor model that was originally proposed in [6] for use in the forthcoming rules on capital charges.

We consider a portfolio loss variable L_n that is defined by

$$L_n = L_n(u_1, \ldots, u_n) = \sum_{i=1}^{n} u_i \, \mathbf{1}_{\{\sqrt{\rho_i}\, X + \sqrt{1-\rho_i}\, \xi_i \leq c_i\}}, \tag{1}$$

where $u_i \geq 0$, $i = 1, \ldots, n$, denotes the weight or the exposure of asset i in the portfolio, $0 < \rho_i < 1$ and $c_i \geq 0$, $i = 1, \ldots, n$, are constants, and $X, \xi_1, \ldots, \xi_n$ are independent random variables with continuous distributions. The constants c_i are called *default thresholds*. They have to be calibrated in order to fix the probabilities of default of the assets. The random variable X is interpreted as the change of an economic factor that influences all the assets in the

portfolio but to different extents. The so-called *asset correlation* ρ_i measures the degree of the i-th asset's exposure to the systematic risk expressed by X. The random variables ξ_i are assumed to model the idiosyncratic (or specific) risk of the assets.

Equation (1) implies the following representation for the conditional variance of the loss L_n given X

$$\mathrm{var}[L_n \,|\, X = x] \;=\; \sum_{i=1}^{n} u_i^2 \,\mathrm{P}\big[\xi_i \leq \tfrac{c_i - \sqrt{\rho_i}\,x}{\sqrt{1-\rho_i}}\big] \,\big(1 - \mathrm{P}\big[\xi_i \leq \tfrac{c_i - \sqrt{\rho_i}\,x}{\sqrt{1-\rho_i}}\big]\big). \qquad (2)$$

We assume that the probabilities of default are not too small and that the correlations with the economic factor are not too high, i.e. in precise terms that $\inf_i c_i > -\infty$ and $\sup_i \rho_i < 1$. Then from (2) it follows that in case of independent, identically distributed $\xi_1, \xi_2, \ldots$ we have

$$\lim_{n\to\infty} \mathrm{E}\big[\mathrm{var}[L_n \,|\, X]\big] = 0 \qquad (3a)$$

if and only if[1]

$$\lim_{n\to\infty} \sum_{i=1}^{n} u_i^2 = 0. \qquad (3b)$$

Since

$$\mathrm{var}[L_n] \;=\; \mathrm{E}\big[\mathrm{var}[L_n \,|\, X]\big] + \mathrm{var}\big[\mathrm{E}[L_n \,|\, X]\big], \qquad (4)$$

the conditional expectation $\mathrm{E}[L_n \,|\, X]$ appears to be a natural approximation of L_n as soon as (3b) is fulfilled, i.e. as soon as the concentrations in the portfolio are not too big. Indeed, the approximation

$$L_n \;\approx\; \mathrm{E}[L_n \,|\, X] \qquad (5)$$

is fundamental for the Basel II approach to credit risk capital charges. For $\alpha \in (0,1)$ and any random variable Y, define the α-quantile (or the Value-at-Risk (VaR)) of Y by

$$q_\alpha(Y) \;=\; \mathrm{VaR}_\alpha(Y) \;=\; \inf\big\{y \in \mathbb{R} : \mathrm{P}[Y \leq y] \geq \alpha\big\}. \qquad (6)$$

Note that

$$\mathrm{E}[L_n \,|\, X = x] \;=\; \sum_{i=1}^{n} u_i \,\mathrm{P}\big[\xi_i \leq \tfrac{c_i - \sqrt{\rho_i}\,x}{\sqrt{1-\rho_i}}\big]. \qquad (7)$$

Since the right-hand side of (7) is a decreasing function in x, one then deduces from (5) that

$$q_\alpha(L_n) \;\approx\; \sum_{i=1}^{n} u_i \,\mathrm{P}\big[\xi_i \leq \tfrac{c_i - \sqrt{\rho_i}\,q_{1-\alpha}(X)}{\sqrt{1-\rho_i}}\big]. \qquad (8a)$$

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[1] Of course, here we admit an additional dependence of u_i on n, i.e. $u_i = u_{i,n}$.

Assuming that the ξ_i are all standard normally distributed then yields

$$q_\alpha(L_n) \approx \sum_{i=1}^{n} u_i \, \Phi\left(\frac{c_i - \sqrt{\rho_i}\, q_{1-\alpha}(X)}{\sqrt{1-\rho_i}}\right), \tag{8b}$$

where Φ denotes the standard normal distribution function. The linearity of the right-hand side of (8b) in the vector $(u_1, \ldots, u_n)$ suggests the choice of

$$\text{Basel II charge}(i) \;=\; u_i \, \Phi\left(\frac{c_i - \sqrt{\rho_i}\, q_{1-\alpha}(X)}{\sqrt{1-\rho_i}}\right) \tag{9}$$

as the capital requirement of asset i in the portfolio with the loss variable L_n. Up to an adjustment for the maturity of the loan which can be neglected in the context of this paper, (9) is just the form of the risk weight functions that was provided by the BCBS in [2].

3   Calculating Risk Contributions with the Semi-Asymptotic Approach

3.1   Definition of semi-asymptotic capital charges

In the following we review the approach by [5] for the definition of the risk contributions of single credit assets if risk is measured with Value-at-Risk (or just a quantile at fixed level). However, we consider here the risk measure Conditional Value-at-Risk which turned out to be more attractive from a conceptual point of view. We call our approach semi-asymptotic because, in contrast to Basel II where all exposures are assumed to be infinitely small, we keep one exposure fixed and let the others tend to infinitely small size.

We consider here a special case of (1) where $\rho_1 = \tau$, $c_1 = a$ but $\rho_i = \rho$ and $c_i = c$ for $i > 1$, and $\sum_{i=1}^{n} u_i = 1$. Additionally, we assume that $u_1 = u$ is a constant for all n but that $u_2, u_3, \ldots$ fulfills (3b).

In this case, the portfolio loss can be represented by

$$L_n(u, u_2, \ldots, u_n) \;=\; u \, \mathbf{1}_{\{\sqrt{\tau} X + \sqrt{1-\tau}\, \xi \leq a\}} +$$
$$(1-u) \sum_{i=2}^{n} u_i \, \mathbf{1}_{\{\sqrt{\rho} X + \sqrt{1-\rho}\, \xi_i \leq c\}}, \tag{10}$$

with $\sum_{i=2}^{n} u_i = 1$. Transition to the limit for $n \to \infty$ in (10) leads to the *semi-asymptotic* percentage loss function

$$L(u) \;=\; u \, \mathbf{1}_D + (1-u) \, Y \tag{11}$$

with $D = \{\sqrt{\tau} X + \sqrt{1-\tau}\, \xi \leq a\}$ and $Y = \mathrm{P}\left[\xi \leq \frac{c - \sqrt{\rho}\, x}{\sqrt{1-\rho}}\right]\Big|_{x=X}$. Of course, a natural choice for τ might be $\tau = \rho$, the mean portfolio asset correlation.

For $\alpha \in (0,1)$ and any random variable Z, define the the Conditional Value-at-Risk (CVaR) (or Expected Shortfall, see [1]) at level α of Z by

$$\mathrm{CVaR}_\alpha(Z) = \mathrm{E}\big[Z \,|\, Z \geq q_\alpha(Z)\big]. \tag{12a}$$

As by (11) we have

$$\mathrm{CVaR}_\alpha(L(u)) = u\,\mathrm{P}[D \,|\, L(u) \geq q_\alpha(L(u))]$$
$$+ (1-u)\,\mathrm{E}\big[Y \,|\, L(u) \geq q_\alpha(L(u))\big], \quad (12\mathrm{b})$$

the following definition is rather near at hand.

The quantity

$$u\,\mathrm{P}[D \,|\, L(u) \geq q_\alpha(L(u))] \tag{13}$$

is called semi-asymptotic CVaR capital charge *(at level α) of the loan with exposure u (as percentage of total portfolio exposure) and default event D as in (11).*

The capital charges we suggest in Definition (13) have to be calculated separately, i.e. for each asset an own model of type (11) has to be regarded. This corresponds to a bottom-up approach since the total capital requirement for the portfolio is determined by adding up all the capital charges of the assets. Note that the capital charges of Definition (13) are not portfolio invariant in the sense of [6]. However, in contrast to the portfolio invariant charges, the semi-asymptotic charges take into account not only correlation but also concentration effects. In particular, their dependence on the exposure u is not merely linear since also the factor $\mathrm{P}[D \,|\, L(u) \geq q_\alpha(L(u))]$ depends upon u. Definition (13) is in line with the general definition of risk contributions (cf. [8], [11]) since (11) can be considered a two-assets portfolio model.

3.2   Calculation of semi-asymptotic capital charges

If F_0 and F_1 denote the conditional distribution functions of Y given $\mathbf{1}_D = 0$ and $\mathbf{1}_D = 1$ respectively, the distribution function of $L(u)$ is given by

$$\mathrm{P}[L(u) \leq z] = p\,F_1\Big(\frac{z-u}{1-u}\Big) + (1-p)\,F_0\Big(\frac{z}{1-u}\Big), \tag{14}$$

where $p = \mathrm{P}[D]$ is the *default probability* of the loan under consideration. By means of (14), the quantile $q_\alpha(L(u))$ can be numerically computed. For the conditional probability which is part of Definition (13), we obtain

$$\mathrm{P}[D \,|\, L(u) \geq z] = \frac{p\left(1 - F_1\big(\frac{z-u}{1-u}\big)\right)}{\mathrm{P}[L(u) \geq z]}. \tag{15}$$

Denote by $\Phi_2(\cdot, \cdot; \theta)$ the distribution function of the bivariate standard normal distribution with correlation θ. If we assume that X and ξ are independent

and both standard normally distributed, we obtain $p = \Phi(a)$, and can derive for the conditional distribution functions from (14) that

$$F_1(z) = \begin{cases} 1 - p^{-1}\, \Phi_2\big(a, \frac{c - \sqrt{1-\rho}\, \Phi^{-1}(z)}{\sqrt{\rho}}; \sqrt{\tau}\big), & z \in (0,1) \\ 0, & \text{otherwise}, \end{cases} \tag{16a}$$

and

$$F_0(z) = \begin{cases} (1-p)^{-1}\, \Phi_2\big(-a, -\frac{c - \sqrt{1-\rho}\, \Phi^{-1}(z)}{\sqrt{\rho}}; \sqrt{\tau}\big), & z \in (0,1) \\ 0, & \text{otherwise}. \end{cases} \tag{16b}$$

Let, similarly to the case of (16a) and (16b), $\Phi_3(\cdot, \cdot, \cdot; \Sigma)$ denote the distribution function of the tri-variate standard normal distribution with correlation matrix Σ. Define the function g by

$$g(x, \beta, z) = \frac{x - \sqrt{1-\beta}\, \Phi^{-1}(z)}{\sqrt{\beta}} \tag{17a}$$

and the correlation matrix $\Sigma_{\rho,\tau}$ by

$$\Sigma_{\rho,\tau} = \begin{pmatrix} 1 & \sqrt{\rho\tau} & \sqrt{\rho} \\ \sqrt{\rho\tau} & 1 & \sqrt{\tau} \\ \sqrt{\rho} & \sqrt{\tau} & 1 \end{pmatrix}. \tag{17b}$$

Then, in order to arrive at $\mathrm{CVaR}_\alpha(L(u))$, the conditional expectation of Y given $\{L(u) \geq z\}$ from (12b) can be calculated according to

$$\mathrm{E}\big[Y \mid L(u) \geq z\big] = \mathrm{P}[L(u) \geq z]^{-1}\Big(\Phi_3\big(c, a, g(c, \rho, \tfrac{z-u}{1-u}); \Sigma_{\rho,\tau}\big) +$$
$$\Phi_2\big(c, g(c, \rho, \tfrac{z}{1-u}); \sqrt{\rho}\big) - \Phi_3\big(c, a, g(c, \rho, \tfrac{z}{1-u}); \Sigma_{\rho,\tau}\big)\Big). \tag{17c}$$

4   Numerical Example

We illustrate the previous results by a numerical example. In our focus is a portfolio that is driven by systematic risk only (the variable Y in (11)) and enlarge this portfolio with an additional loan (the indicator $\mathbf{1}_D$ in (11)).

In our example, the portfolio modeled by Y has a quite moderate credit standing which is expressed by its expected loss $\mathrm{E}[Y] = 0.025 = \Phi(c)$. By choosing $\rho = 0.1$ as asset correlation we arrive at a portfolio with a rather strong exposure to systematic risk. For the sake of simplicity we choose $\tau = \rho$, i.e. the exposure to systematic risk of the additional loan is identical to the exposure of the existing portfolio. However, we assume that the additional loan enjoys a quite high credit-worthiness as we set $p = \mathrm{P}[D] = 0.002 = \Phi(a)$.

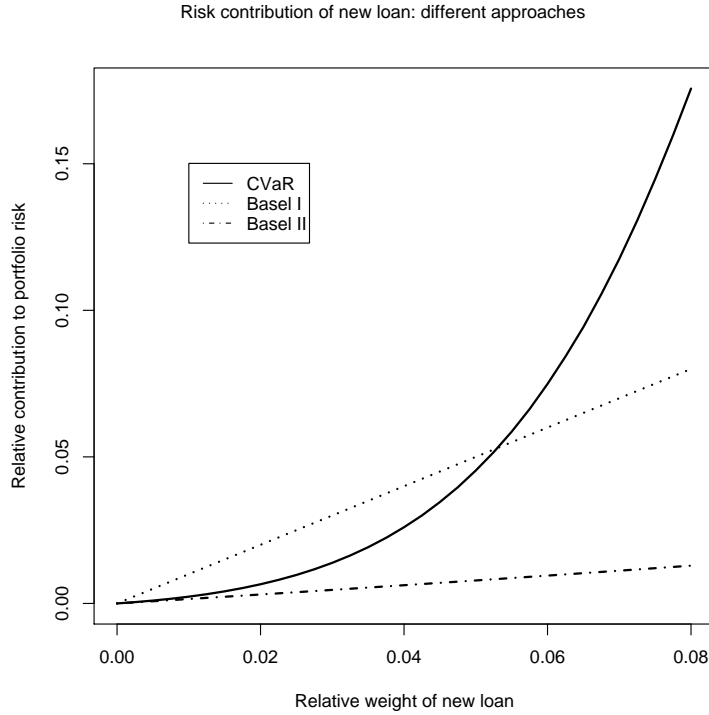Risk contribution of new loan: different approaches

Figure 1: Relative risk contribution of new loan as function of the relative weight of the new loan. Comparison of contribution to true Conditional Value-at-Risk (CVaR), and the contributions according to the Basel II and Basel I Accords.

Figure 1 illustrates the relative contribution of the new loan to the risk of the portfolio loss variable $L(u)$. The contribution is expressed as a function of the relative weight u of the new loan in the portfolio and calculated according to three different methods. The first of the depicted methods relates to the relative contribution to true portfolio CVaR at level $\alpha = 99.9\%$, defined as the ratio of the contribution to CVaR according to Definition (13) and portfolio CVaR, i.e. the function

$$u \;\mapsto\; \frac{u\,\mathrm{P}[D \mid L(u) \geq q_\alpha(L(u))]}{\mathrm{CVaR}_\alpha(L(u))}, \tag{18}$$

where the conditional probability has to be evaluated by means of (15) and (16a). In the denominator, CVaR is calculated according to (12b) and (17c). Moreover, curves are drawn for the Basel II approach, i.e. the function

$$u \;\mapsto\; \frac{u\,\Phi\left(\frac{a - \sqrt{\tau}\,q_{1-\alpha}(X)}{\sqrt{1-\tau}}\right)}{u\,\Phi\left(\frac{a - \sqrt{\tau}\,q_{1-\alpha}(X)}{\sqrt{1-\tau}}\right) + (1-u)\,\Phi\left(\frac{c - \sqrt{\rho}\,q_{1-\alpha}(X)}{\sqrt{1-\rho}}\right)}, \tag{19}$$

and the Basel I approach. The latter approach just entails the diagonal as
risk contribution curve since it corresponds to purely volume-oriented capital
allocation.

Note that in Figure 1 the true CVaR curve intersects the diagonal (Basel I
curve) just at the relative weight u^* that corresponds to the minimum risk
portfolio $L(u^*)$. The Basel II curve differs strongly from the true contribution
curve and is completely situated below the diagonal. This fact could yield the
misleading impression that an arbitrarily high exposure to the additional loan
still improves the risk of the portfolio. However, as the true CVaR curve in
Figure 1 shows, the diversification effect from pooling with the new loan stops
at 5.8% relative weight.

To sum up, it can be said that the example shows a shortcoming of the
new Basel II capital requirement rules as they are not sensitive to concentra-
tions. In addition, the example presents an intuitive bottom-up approach for
calculating contributions that is sensitive to correlation as well as to concen-
trations and avoids time-consuming simulations.

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